2 Vector spaces with additional structure

In the following $\mathbb{K}$ denotes a field which might be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. Let V be a vector space over $\mathbb{K}$. A subset A of V is called *balanced* iff for all $v \in A$ and all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$ the vector λv is contained in A. A subset A of V is called *convex* iff for all $x, y \in V$ and $t \in [0,1]$ the vector $(1-t)x + ty$ is in A. Let A be a subset of V. Consider the smallest subset of V which is convex and which contains A. This is called the *convex hull* of A, denoted $\mathrm{conv}(A)$.

Proposition 2.2. *(a) Intersections of balanced sets are balanced. (b) The sum of two balanced sets is balanced. (c) A scalar multiple of a balanced set is balanced.*

Proof. **Exercise.** □

Proposition 2.3. *Let V be vector space and A a subset. Then*

$$\mathrm{conv}(A) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : \lambda_i \in [0,1], x_i \in A, \sum_{i=1}^{n} \lambda_i = 1 \right\}.$$

Proof. **Exercise.** □

We denote the space of linear maps between a vector space V and a vector space W by $\mathrm{L}(V, W)$.

2.1 Topological vector spaces

Definition 2.4. A set V that is equipped both with a vector space structure over $\mathbb{K}$ and a topology is called a *topological vector space (tvs)* iff the vector addition $+ : V \times V \to V$ and the scalar multiplication $\cdot : \mathbb{K} \times V \to V$ are both continuous. (Here the topology on $\mathbb{K}$ is the standard one.)

Proposition 2.5. *Let V be a tvs, $\lambda \in \mathbb{K} \setminus 0$, $w \in V$. The maps $V \to V : v \mapsto \lambda v$ and $V \to V : v \mapsto v + w$ are automorphisms of V as a tvs. In particular, the topology $\mathcal{T}$ of V is invariant under rescalings and translations: $\lambda\mathcal{T} = \mathcal{T}$ and $\mathcal{T} + w = \mathcal{T}$. In terms of filters of neighborhoods, $\lambda\mathcal{N}_v = \mathcal{N}_{\lambda v}$ and $\mathcal{N}_v + w = \mathcal{N}_{v+w}$ for all $v \in V$.*

Proof. It is clear that non-zero scalar multiplication and translation are vector space automorphisms. To see that they are also continuous use Proposition 1.16. The inverse maps are of the same type hence also continuous. Thus we have homeomorphisms. The scale- and translation invariance of the topology follows. □

Note that this implies that the topology of a tvs is completely determined by the filter of neighborhoods of one of its points, say 0.

Definition 2.6. Let V be a tvs and U a subset. U is called *bounded* iff for every neighborhood W of 0 there exists $\lambda \in \mathbb{R}^+$ such that $U \subseteq \lambda W$.

Remark: Changing the allowed range of λ in the definition of boundedness from $\mathbb{R}^+$ to $\mathbb{K}$ leads to an equivalent definition, i.e., is not weaker. However, the choice of $\mathbb{R}^+$ over $\mathbb{K}$ is more convenient in certain applications.

Proposition 2.7. *Let V be a tvs. Then:*

1. *Every point set is bounded.*
2. *Every neighborhood of* 0 *contains a balanced subneighborhood of* 0.
3. *Let U be a neighborhood of* 0. *Then there exists a subneighborhood W of* 0 *such that $W + W \subseteq U$.*

Proof. We start by demonstrating Property 1. Let $x \in V$ and U some open neighborhood of 0. Then $Z := \{(\lambda, y) \in \mathbb{K} \times V : \lambda y \in U\}$ is open by continuity of multiplication. Also $(0, x) \in Z$ so that by the product topology there exists an $\epsilon > 0$ and an open neighborhood W of x in V such that $B_\epsilon(0) \times W \subseteq Z$. In particular, there exists $\mu > 0$ such that $\mu x \in U$, i.e., $\{x\} \subseteq \mu^{-1} U$ as desired.

We proceed to Property 2. Let U be an open neighborhood of 0. By continuity $Z := \{(\lambda, x) \in \mathbb{K} \times V : \lambda x \in U\}$ is open. By the product topology, there are open neighborhoods X of $0 \in \mathbb{K}$ and W of $0 \in V$ such that $X \times W \subseteq Z$. Thus, $X \cdot W \subseteq U$. Now X contains an open ball of some radius $\epsilon > 0$ around 0 in $\mathbb{K}$. Set $Y := B_\epsilon(0) \cdot W$. This is an (open) neighborhood of 0 in V, it is contained in U and it is balanced.

We end with Property 3. Let U be an open neighborhood of 0. By continuity $Z := \{(x, y) \in V \times V : x + y \in U\}$ is open. By the product topology, there are open neighborhoods W_1 and W_2 of 0 such that $W_1 \times W_2 \subseteq Z$. This means $W_1 + W_2 \subseteq U$. Now define $W := W_1 \cap W_2$. □

Proposition 2.8. *Let V be a vector space and $\mathcal{F}$ a filter on V. Then $\mathcal{F}$ is the filter of neighborhoods of* 0 *for a compatible topology on V iff* 0 *is contained in every element of $\mathcal{F}$ and $\lambda \mathcal{F} = \mathcal{F}$ for all $\lambda \in \mathbb{K} \setminus \{0\}$ and $\mathcal{F}$ satisfies the properties of Proposition 2.7.*

Proof. It is already clear that the properties in question are necessary for $\mathcal{F}$ to be the filter of neighborhoods of 0 of V. It remains to show that they are sufficient. If $\mathcal{F}$ is to be the filter of neighborhoods of 0 then, by translation invariance, $\mathcal{F}_x := \mathcal{F} + x$ must be the filter of neighborhoods of the point x. We show that the family of filters $\{\mathcal{F}_x\}_{x \in V}$ does indeed define a topology on V. To this end we will use Proposition 1.10. Property 1 is satisfied by assumption. It remains to show Property 2. By translation invariance it will be enough to consider $x = 0$. Suppose $U \in \mathcal{F}$. Using Property 3 of Proposition 2.7 there is $W \in \mathcal{F}$ such that $W + W \subseteq U$. We claim that Property 2 of Proposition 1.10 is now satisfied with this choice of W. Indeed, let $y \in W$ then $y + W \in \mathcal{F}_y$ and $y + W \subseteq U$ so $U \in \mathcal{F}_y$ as required.

We proceed to show that the topology defined in this way is compatible with the vector space structure. Take an open set $U \subseteq V$ and consider its preimage $Z = \{(x, y) \in V \times V : x + y \in U\}$ under vector addition. Take some point $(x, y) \in Z$. $U - x - y$ is an open neighborhood of 0. By Property 3 of Proposition 2.7 there is an open neighborhood W of 0 such that $W + W = U - x - y$, i.e., $(x + W) + (y + W) \subseteq U$. But $x + W$ is an open neighborhood of x and $y + W$ is an open neighborhood of y so $(x + W) \times (y + W)$ is an open neighborhood of (x, y) in $V \times V$ contained in Z. Hence vector addition is continuous.

We proceed to show continuity of scalar multiplication. Consider an open set $U \subseteq V$ and consider its preimage $Z = \{(\lambda, x) \in \mathbb{K} \times V : \lambda x \in U\}$ under scalar multiplication. Take some point $(\lambda, x) \in Z$. $U - \lambda x$ is an open neighborhood of 0 in V. By Property 3 of Proposition 2.7 there is an open neighborhood W of 0 such that $W + W = U - \lambda x$. By Property 2 of Proposition 2.7 there exists a balanced subneighborhood X of W. By Property 1 of Proposition 2.7 (boundedness of points) there exists $\epsilon > 0$ such that $\epsilon x \in X$. Since X is balanced, $B_\epsilon(0) \cdot x \subseteq X$. Now define $Y := (\epsilon + |\lambda|)^{-1} X$. Note that scalar multiples of (open) neighborhoods of 0 are (open) neighborhoods of 0 by assumption. Hence Y is open since X is. Thus $B_\epsilon(\lambda) \times (x + Y)$ an open neighborhood of (λ, x) in $\mathbb{K} \times V$. We claim that it is contained in Z. First observe that since X is balanced, $B_\epsilon(0) \cdot x \subseteq X$. Similarly, we have $B_\epsilon(\lambda) \cdot Y \subseteq B_{\epsilon + |\lambda|}(0) \cdot Y = B_1(0) \cdot X \subseteq X$. Thus we have $B_\epsilon(0) \cdot x + B_\epsilon(\lambda) \cdot Y \subseteq X + X \subseteq W + W \subseteq U - \lambda x$. But this implies $B_\epsilon(\lambda) \cdot (x + W) \subseteq U$ as required. □

Proposition 2.9. *In a tvs every neighborhood of* 0 *contains a closed and balanced subneighborhood.*

Proof. Let U be a neighborhood of 0. By Proposition 2.7.3 there exists a subneighborhood $W \subseteq U$ such that $W + W \subset U$. By Proposition 2.7.2 there exists a balanced subneighborhood $X \subseteq W$. Let $Y := \overline{X}$. Then, Y is obviously a closed neighborhood of 0. Also Y is balanced, since for $y \in Y$ and $\lambda \in \mathbb{K}$ with $0 < |\lambda| \leq 1$ we have $\lambda y \in \lambda\overline{X} = \overline{\lambda X} \subseteq \overline{X} = Y$. Finally, let $y \in Y = \overline{X}$. Any neighborhood of y must intersect X. In particular, $y + X$ is such a neighborhood. Thus, there exist $x \in X$, $z \in X$ such that $x = y + z$, i.e., $y = x - z \in X - X = X + X \subseteq U$. So, $Y \subseteq U$. □

Proposition 2.10. *(a) Subsets of bounded sets are bounded. (b) Intersections and finite unions of bounded sets are bounded. (c) The closure of a bounded set is bounded. (d) The sum of two bounded sets is bounded. (e) A scalar multiple of a bounded set is bounded. (f) Compact sets are bounded.*

Proof. **Exercise.** □

Let A, B be topological vector spaces. We denote the space of maps from A to B that are linear and continuous by $\mathrm{CL}(A, B)$.

Definition 2.11. Let A, B be tvs. A linear map $f : A \to B$ is called *bounded* iff there exists a neighborhood U of 0 in A such that $f(U)$ is bounded. A linear map $f : A \to B$ is called *compact* iff there exists a neighborhood U of 0 in A such that $\overline{f(U)}$ is compact.

Let A, B be tvs. We denote the space of maps from A to B that are linear and bounded by $\mathrm{BL}(A, B)$.

Proposition 2.12. *Let A, B be tvs and $f \in \mathrm{L}(A, B)$. (a) f is continuous iff the preimage of any neighborhood of 0 in B is a neighborhood of 0 in A. (b) If f is continuous it maps bounded sets to bounded sets. (c) If f is bounded then f is continuous, i.e., $\mathrm{BL}(A, B) \subseteq \mathrm{CL}(A, B)$. (d) If f is compact then f is bounded.*

Proof. **Exercise.** □

A useful property for a topological space is the Hausdorff property, i.e., the possibility to separate points by open sets. It is not the case that a tvs is automatically Hausdorff. However, the way in which a tvs may be non-Hausdorff is severly restricted. Indeed, we shall see int the following that a tvs may be split into a part that is Hausdorff and another one that is maximally non-Hausdorff in the sense of carrying the trivial topology.

Proposition 2.13. *Let V be a tvs and $C \subseteq V$ a vector subspace. Then, the closure $\overline{C}$ of C is also a vector subspace of V.*

Proof. **Exercise.**[Hint: Use Proposition 1.32.] □

Proposition 2.14. *Let V be a tvs. The closure of $\{0\}$ in V coincides with the intersection of all neighborhoods of* 0*. Moreover, V is Hausdorff iff $\overline{\{0\}} = \{0\}$.*

Proof. **Exercise.** □

Proposition 2.15. *Let V be a tvs and $C \subseteq V$ a vector subspace. Then the quotient space V/C is a tvs. Moreover, V/C is Hausdorff iff C is closed in V.*

Proof. **Exercise.** □

Thus, for a tvs V the exact sequence

$$0 \to \overline{\{0\}} \to V \to V/\overline{\{0\}} \to 0$$

describes how V is composed of a Hausdorff piece $V/\overline{\{0\}}$ and a piece $\overline{\{0\}}$ with trivial topology. We can express this decomposition also in terms of a direct sum, as we shall see in the following.

A (vector) subspace of a tvs is a tvs with the subset topology. Let A and B be tvs. Then the direct sum $A \oplus B$ is a tvs with the product topology. Note that as subsets of $A \oplus B$, both A and B are closed.

Definition 2.16. Let V be a tvs and A a subspace. Then another subspace B of A in V is called a *topological complement* iff $V = A \oplus B$ as tvs (i.e., as vector spaces and as topological spaces). A is called *topologically complemented* if such a topological complement B exists.

Note that algebraic complements (i.e., complements merely with respect to the vector space structure) always exist (using the Axiom of Choice). However, an algebraic complement is not necessarily a topological one. Indeed, there are examples of subspaces of tvs that have no topological complement.

Proposition 2.17 (Structure Theorem for tvs)**.** *Let V be a tvs and B an algebraic complement of $\overline{\{0\}}$ in V. Then B is also a topological complement of $\overline{\{0\}}$ in V. Moreover, B is canonically isomorphic to $V/\overline{\{0\}}$ as a tvs.*

Proof. **Exercise.** □

We conclude that every tvs is a direct sum of a Hausdorff tvs and a tvs with the trivial topology.

2.2 Metrizable vector spaces

In this section we consider *metrizable vector spaces* (mvs), i.e., tvs that admit a metric compatible with the topology.

Definition 2.18. A metric on a vector space V is called *translation-invariant* iff $d(x+a,y+a)=d(x,y)$ for all $x,y,a\in V$. A translation-invariant metric on a vector space V is called *balanced* iff its open balls around the origin are balanced.

As we shall see it will be possible to limit ourselves to balanced translation-invariant metrics on mvs. Moreover, these can be conveniently described by pseudo-norms.

Definition 2.19. Let V be a vector space over $\mathbb{K}$. Then a map $V\to\mathbb{R}^+_0 : x\mapsto\|x\|$ is called a *pseudo-norm* iff it satisfies the following properties:

1. For all $\lambda\in\mathbb{K}$, $|\lambda|\le 1$ implies $\|\lambda x\|\le\|x\|$ for all $x\in V$.
2. For all $x,y\in V:\ \|x+y\|\le\|x\|+\|y\|$.
3. $\|x\|=0$ iff $x=0$.

Proposition 2.20. *There is a one-to-one correspondence between pseudo-norms and balanced translation invariant metrics on a vector space via* $d(x,y):=\|x-y\|$.

Proof. **Exercise.** □

Proposition 2.21. *Let V be a vector space. The topology generated by a pseudo-norm on V is compatible with the vector space structure iff for every $x\in V$ and $\epsilon>0$ there exists $\lambda\in\mathbb{R}^+$ such that $x\in\lambda B_\epsilon(0)$.*

Proof. Assume we are given a pseudo-norm on V that induces a compatible topology. It is easy to see that the stated property of the pseudo-norm then follows from Property 1 in Proposition 2.7 (boundedness of points).

Conversely, suppose we are given a pseudo-norm on V with the stated property. We show that the filter $\mathcal{N}_0$ of neighborhoods of 0 defined by the pseudo-norm has the properties required by Proposition 2.8 and hence defines a compatible topology on V. Firstly, it is already clear that every $U\in\mathcal{N}_0$ contains 0. We proceed to show that $\mathcal{N}_0$ is scale invariant. It is enough to show that for $\epsilon>0$ and $\lambda\in\mathbb{K}\setminus\{0\}$ the scaled ball $\lambda B_\epsilon(0)$ is open. Choose a point $\lambda x\in\lambda B_\epsilon(0)$. Take $\delta>0$ such that $|x|<\epsilon+\delta$. Then $B_\delta(0)+x\subseteq B_\epsilon(0)$. If $|\lambda|\ge 1$ then $B_\delta(\lambda x)=B_\delta(0)+\lambda x\subseteq\lambda B_\delta(0)+\lambda x\subseteq$

$\lambda B_\epsilon(0)$ showing that $\lambda B_\epsilon(0)$ is open. Assume now that $|\lambda| \le 1$ and choose $n \in \mathbb{N}$ such that $2^{-n} \le |\lambda|$. Observe that the triangle inequality implies $B_{2^{-n}\delta}(0) \subseteq 2^{-n} B_\delta(0)$ (for arbitrary δ and n in fact). Hence $B_{2^{-n}\delta}(\lambda x) = B_{2^{-n}\delta}(0) + \lambda x \subseteq \lambda B_\delta(0) + \lambda x \subseteq \lambda B_\epsilon(0)$ showing that $\lambda B_\epsilon(0)$ is open.

It now remains to show the properties of $\mathcal{N}_0$ listed in Proposition 2.7. As for Property 3, we may take U to be an open ball of radius ϵ around 0 for some $\epsilon > 0$. Define $W := B_{\epsilon/2}(0)$ Then $W + W \subseteq U$ follows from the triangle inequality. Concerning Property 2 we simple notice that open balls are balanced by construction. The only property that is not automatic for a pseudo-norm and does require the stated condition is Property 1 (boundedness of points). The equivalence of the two is easy to see. □

Theorem 2.22. *A Hausdorff tvs V is metrizable iff it is first-countable, i.e., iff there exists a countable base for the filter of neighborhoods of* 0*. Moreover, if V is metrizable it admits a compatible pseudo-norm.*

Proof. It is clear that metrizability implies the existence of a countable base of $\mathcal{N}_0$. For example, the sequence of balls $\{B_{1/n}(0)\}_{n\in\mathbb{N}}$ provides such a base. Conversely, suppose that $\{U_n\}_{n\in\mathbb{N}}$ is a base of the filter of neighborhoods of 0 such that all U_n are balanced and $U_{n+1} + U_{n+1} \subseteq U_n$. (Given an arbitrary countable base of $\mathcal{N}_0$ we can always produce another one with the desired properties.) Now for each finite subset H of $\mathbb{N}$ define $U_H := \sum_{n\in H} U_n$ and $\lambda_H := \sum_{n\in H} 2^{-n}$. Note that each U_H is a balanced neighborhood of 0. Define now the function $V \to \mathbb{R}^+_0 : x \mapsto \|x\|$ by

$$\|x\| := \inf_H \{\lambda_H | x \in U_H\}$$

if $x \in U_H$ for some H and $\|x\| = 1$ otherwise. We proceed to show that $\|\cdot\|$ defines a pseudo-norm and generates the topology of V.

Fix $x \in V$ and $\lambda \in \mathbb{K}$ with $|\lambda| \le 1$. Since U_H is balanced for each H, λx is contained at least in the same sets U_H as x. Because the definition of $\|\cdot\|$ uses an infimum, $\|\lambda x\| \le \|x\|$. This confirms Property 1 of Definition 2.19.

To show the triangle inequality (Property 2 of Definition 2.19) we first note that for finite subsets H, K of $\mathbb{N}$ with the property $\lambda_H + \lambda_K < 1$ there is another unique finite subset L of $\mathbb{N}$ such that $\lambda_L = \lambda_H + \lambda_K$. Furthermore, $U_H + U_K \subseteq U_L$ in this situation. Now, fix $x, y \in V$. If $\|x\| + \|y\| \ge 1$ the triangle inequality is trivial. Otherwise, we can find $\epsilon > 0$ such that $\|x\| + \|y\| + 2\epsilon < 1$. We now fix finite subsets H, K of $\mathbb{N}$ such that $x \in U_H$, $y \in U_K$ while $\lambda_H < \|x\| + \epsilon$ and $\lambda_K < \|y\| + \epsilon$. Let L be the finite subset of $\mathbb{N}$ such that $\lambda_L = \lambda_H + \lambda_K$. Then $x + y \in U_L$ and hence $\|x + y\| \le \lambda_L =$

$\lambda_H + \lambda_K < \|x\| + \|y\| + 2\epsilon$. Since the resulting inequality holds for any $\epsilon > 0$ we must have $\|x + y\| \leq \|x\| + \|y\|$ as desired.

If $x = 0$ clearly $\|x\| = 0$. Conversely, if $x \neq 0$ the Hausdorff property of V implies that there exists some $n \in \mathbb{N}$ such that $U_n = U_{\{n\}}$ does not contain x. Since $\lambda_H \leq \lambda_K$ implies $U_H \subseteq U_K$ this means $\|x\| \geq 2^{-n}$. This confirms Property 3 of Definition 2.19.

It remains to show that the pseudo-norm generates the topology of the tvs. Since the topology generated by the pseudo-norm as well as that of the tvs are translation invariant, it is enough to show that the open balls around 0 of the pseudo-norm form a base of the filter of neighborhoods of 0 in the topology of the tvs. Let $n \in \mathbb{N}$ and $\epsilon > 0$. Clearly $B_{2^{-n}}(0) \subseteq U_n \subseteq B_{2^{-n}+\epsilon}(0)$. By the arbitraryness of ϵ this implies $U_n = B_{2^{-n}}(0)$. But $\{U_n\}_{n\in\mathbb{N}}$ is a base of the filter of neighborhoods of 0 by assumption. And if the balls $\{B_{2^{-n}}(0)\}_{n\in\mathbb{N}}$ form such a base then clearly all balls around 0 also form a base. This completes the proof. □

Proposition 2.23. *Let V be a mvs with pseudo-norm. Let $r > 0$ and $0 < \mu \leq 1$. Then, $B_{\mu r}(0) \subseteq \mu B_r(0)$.*

Proof. **Exercise.** □

Proposition 2.24. *Let V, W be mvs with compatible metrics and $f \in \mathrm{L}(V, W)$. (a) f is continuous iff for all $\epsilon > 0$ there exists $\delta > 0$ such that $f(B^V_\delta(0)) \subseteq B^W_\epsilon(0)$. (b) f is bounded iff there exists $\delta > 0$ such that for all $\epsilon > 0$ there is $\mu > 0$ such that $f(\mu B^V_\delta(0)) \subseteq B^W_\epsilon(0)$.*

Proof. **Exercise.** □

2.3 Normed vector spaces

Definition 2.25. A tvs is called *locally bounded* iff it contains a bounded neighborhood of 0.

Proposition 2.26. *A locally bounded Hausdorff tvs is metrizable.*

Proof. Let V be a locally bounded Hausdorff tvs and U a bounded neighborhood of 0 in V. The sequence $\{U_n\}_{n\in N}$ with $U_n := \frac{1}{n}U$ is the base of a filter $\mathcal{F}$ on V. Take a neighborhood W of 0. By boundedness of U there exists $\lambda \in \mathbb{R}^+$ such that $U \subseteq \lambda W$. Choosing $n \in \mathbb{N}$ with $n \geq \lambda$ we find $U_n \subseteq W$, i.e., $W \in \mathcal{F}$. Hence $\mathcal{F}$ is the filter of neighborhoods of 0 and we have presented a countable base for it. By Theorem 2.22, V is metrizable. □

Proposition 2.27. *Let A, B be a tvs and $f \in \mathrm{CL}(A, B)$. If A or B is locally bounded then f is bounded. Hence,* $\mathrm{CL}(A, B) = \mathrm{BL}(A, B)$ *in this case.*

Proof. **Exercise**. □

Definition 2.28. A tvs is called *locally convex* iff every neighborhood of 0 contains a convex neighborhood of 0.

Definition 2.29. Let V be a vector space over $\mathbb{K}$. Then a map $V \to \mathbb{R}_0^+ : x \mapsto \|x\|$ is called a *norm* iff it satisfies the following properties:

1. $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{K}, x \in V$.
2. For all $x, y \in V : \|x + y\| \leq \|x\| + \|y\|$.
3. $\|x\| = 0 \implies x = 0$.

A norm is a pseudo-norm and hence induces a metric and a topology.

Proposition 2.30. *The topology induced by a norm on a vector space makes it into a tvs.*

Proof. Since a norm is in particular a pseudo-norm we may apply Proposition 2.21. Indeed, let V be a vector space with a norm $\|\cdot\|$. Take $x \in V$ and $\epsilon > 0$. Then $x \in \lambda B_\epsilon(0)$ if we choose $\lambda > 0$ such that $\|x\| < \lambda\epsilon$, satisfying the condition of the Proposition. □

In the following we shall be interested in *normed vector spaces*, i.e., vector spaces equipped with a norm.

Theorem 2.31. *A Hausdorff tvs V is normable iff V is locally bounded and locally convex.*

Proof. Suppose V is a normed vector space. It is easy to see that every ball is bounded and also convex, so in particular, V is locally bounded and locally convex.

Conversely, suppose V is a Hausdorff tvs that is locally bounded and locally convex. Take a bounded neighborhood U_1 of 0 and a convex subneighborhood U_2 of U_1. Now take a balanced subneighborhood U_3 of U_2 and its convex hull $W = \mathrm{conv}(U_3)$. Then W is a balanced, convex and bounded (since $W \subseteq U_2 \subseteq U_1$) neighborhood of 0 in V. Define the *Minkowski functional* $\|\cdot\|_W : V \to \mathbb{R}_0^+$ associated with W as

$$\|x\|_W := \inf\{\lambda \in \mathbb{R}_0^+ : x \in \lambda W\}.$$

We claim that $\|\cdot\|_W$ defines a norm on V that is compatible with its topology. Linearity (Property 1 of Definition 2.29) follows by noticing that balancedness of W implies $\lambda W = |\lambda| W$ for $\lambda \in \mathbb{K}$. The triangle inequality (Property 2 of Definition 2.29) can be seen as follows: Given points $x, y \in V$ we fix some $\epsilon > 0$. Now choose $\mu_x > 0$ and $\mu_y > 0$ such that $\mu_x < \|x\|_W + \epsilon$ and $\mu_y < \|y\|_W + \epsilon$ while $x \in \mu_x W$ and $y \in \mu_y W$. Thus $\mu_x^{-1} x \in W$ and $\mu_y^{-1} y \in W$. Set $t := \mu_x/(\mu_x + \mu_y)$. Convexity of W implies $(x+y)/(\mu_x+\mu_y) = t\mu_x^{-1}x + (1-t)\mu_y^{-1} y \in W$. Hence $x + y \in (\mu_x + \mu_y) W$ and thus $\|x+y\|_W \le \mu_x + \mu_y < \|x\|_W + \|y\|_W + 2\epsilon$. Since ϵ was arbitrary $\|x+y\|_W \le \|x\|_W + \|y\|_W$ follows. We proceed to show definiteness (Property 3 of Definition 2.29). Take $x \neq 0$. By the Hausdorff property there is a neighborhood U of 0 that does not contain x. Since W is bounded there exists $\lambda \in \mathbb{R}^+$ such that $W \subseteq \lambda U$. Hence $\mu W \subseteq \lambda^{-1} W \subseteq U$ for all $\mu \le \lambda^{-1}$ since W is balanced. This means $x \notin \mu W$ for $\mu \le \lambda^{-1}$ and thus $\|x\|_W \ge \lambda^{-1}$.

It remains to show that the topology generated by the norm $\|\cdot\|_W$ coincides with the topology of V. Let U be an open set in the topology of V and $x \in U$. The ball $B_1(0)$ defined by the norm is bounded since $B_1(0) \subseteq W$ and W is bounded. Hence there exists $\lambda \in \mathbb{R}^+$ such that $B_1(0) \subseteq \lambda(U - x)$, i.e., $\lambda^{-1} B_1(0) \subseteq U - x$. But $\lambda^{-1} B_1(0) = B_{\lambda^{-1}}(0)$ by linearity and thus $B_{\lambda^{-1}}(x) \subseteq U$. Hence, U is open in the norm topology as well. Conversely, consider a ball $B_\epsilon(0)$ defined by the norm for some $\epsilon > 0$ and take $x \in B_\epsilon(0)$. Choose $\delta > 0$ such that $\mu_W(x) < \epsilon + \delta$. Observe that $\frac{1}{2} W \subseteq B_1(0)$ and thus by linearity $\frac{\delta}{2} W \subseteq B_\delta(0)$. It follows that $\frac{\delta}{2} W + x \subseteq B_\epsilon(0)$. But $\frac{\delta}{2} W + x$ is a neighborhood of x so it follows that $B_\epsilon(0)$ is open. This completes the proof. □

Proposition 2.32. *Let V be a normed vector space and $U \subseteq V$ a subset. Then, U is bounded iff there exists $c \in \mathbb{R}^+$ such that $\|x\| \le c$ for all $x \in U$.*

Proof. **Exercise**. □

Proposition 2.33. *Let A, B be normed vector spaces and $f \in \mathrm{L}(A, B)$. f is bounded iff there exists $c \in \mathbb{R}^+$ such that $\|f(x)\| \le c\,\|x\|$ for all $x \in A$.*

Proof. **Exercise**. □

2.4 Inner product spaces

As before $\mathbb{K}$ stands for a field that is either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.34. Let V be a vector space over $\mathbb{K}$ and $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{K}$ a map. $\langle\cdot,\cdot\rangle$ is called a *bilinear* (if $\mathbb{K} = \mathbb{R}$) or *sesquilinear* (if $\mathbb{K} = \mathbb{C}$) form iff it satisfies the following properties:

- $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ and
  $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$ for all $u, v, w \in V$.
- $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ and $\langle u, \lambda v\rangle = \overline{\lambda}\langle u, v\rangle$ for all $\lambda \in \mathbb{K}$ and $v \in V$.

$\langle\cdot,\cdot\rangle$ is called *symmetric* (if $\mathbb{K} = \mathbb{R}$) or *hermitian* (if $\mathbb{K} = \mathbb{C}$) iff it satisfies in addition the following property:

- $\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$.

$\langle\cdot,\cdot\rangle$ is called *positive* iff it satisfies in addition the following property:

- $\langle v, v\rangle \geq 0$ for all $v \in V$.

$\langle\cdot,\cdot\rangle$ is called *definite* iff it satisfies in addition the following property:

- If $\langle v, v\rangle = 0$ then $v = 0$ for all $v \in V$.

A map with all these properties is also called a *scalar product* or an *inner product*. V equipped with such a structure is called an *inner product space* or a *pre-Hilbert space*.

Theorem 2.35 (Schwarz Inequality)**.** *Let V be a vector space over $\mathbb{K}$ with a scalar product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{K}$. Then, the following inequality is satisfied:*

$$|\langle v, w\rangle|^2 \leq \langle v, v\rangle\langle w, w\rangle \quad \forall v, w \in V.$$

Proof. By definiteness $\alpha := \langle v, v\rangle \neq 0$ and we set $\beta := -\langle w, v\rangle$. By positivity we have,

$$0 \leq \langle \beta v + \alpha w, \beta v + \alpha w\rangle.$$

Using bilinearity and symmetry (if $\mathbb{K} = \mathbb{R}$) or sesquilinearity and hermiticity (if $\mathbb{K} = \mathbb{C}$) on the right hand side this yields,

$$0 \leq |\langle v, v\rangle|^2\langle w, w\rangle - \langle v, v\rangle|\langle v, w\rangle|^2.$$

(**Exercise.**Show this.) Since $\langle v, v\rangle \neq 0$ we can divide by it and arrive at the required inequality. □

Proposition 2.36. *Let V be a vector space over $\mathbb{K}$ with a scalar product $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{K}$. Then, V is a normed vector space with norm given by $\|v\| := \sqrt{\langle v, v\rangle}$.*

Proof. **Exercise.**Hint: To prove the triangle inequality, show that $\|v+w\|^2 \leq (\|v\|+\|w\|)^2$ can be derived from the Schwarz inequality (Theorem 2.35). □

Proposition 2.37. *Let V be an inner product space. Then, $\forall v, w \in V$,*

$$\langle v,w\rangle = \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2\right) \quad \textit{if} \quad \mathbb{K}=\mathbb{R},$$
$$\langle v,w\rangle = \frac{1}{4}\left(\|v+w\|^2 - \|v-w\|^2 + \mathrm{i}\|v+\mathrm{i}w\|^2 - \mathrm{i}\|v-\mathrm{i}w\|^2\right) \quad \textit{if} \quad \mathbb{K}=\mathbb{C}$$

Proof. **Exercise.** □

Proposition 2.38. *Let V be an inner product space. Then, its scalar product $V \times V \to \mathbb{K}$ is continuous.*

Proof. **Exercise.** □

Theorem 2.39. *Let V be a normed vector space. Then, there exists a scalar product on V inducing the norm iff the parallelogram equality holds,*

$$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2 \quad \forall v, w \in V.$$

Proof. **Exercise.** □

Example 2.40. The spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ are inner product spaces via

$$\langle v,w\rangle := \sum_{i=1}^{n} v_i \overline{w_i},$$

where v_i, w_i are the coefficients with respect to the standard basis.